

# Intersection cohomology of the circle actions<sup>\*</sup>

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## Abstract

A classical result says that a free action of the circle  $\mathbb{S}^1$  on a topological space  $X$  is geometrically classified by the orbit space  $B$  and by a cohomological class  $e \in H^2(B, \mathbb{Z})$ , the Euler class. When the action is not free we have a difficult open question:

II : “Is the space  $X$  determined by the orbit space  $B$  and the Euler class?”

The main result of this work is a step towards the understanding of the above question in the category of unfolded pseudomanifolds. We prove that the orbit space  $B$  and the Euler class determine:

- the intersection cohomology of  $X$ ,
- the real homotopy type of  $X$ .

In this work, we give an answer to the question II in the category of unfolded pseudomanifolds. The object studied are the modelled actions  $\Phi: \mathbb{S}^1 \times X \rightarrow X$ . Here, the total space  $X$  is an unfolded pseudomanifold and the action  $\Phi$  preserves this structure in such a way that the orbit space  $B$  is still an unfolded pseudomanifold.

A priori, the action  $\Phi$  classifies the strata of  $X$  in two types: the mobile strata (containing one-dimensional orbits), and the fixed strata (containing the fixed points). But we see in this work that we need a finer classification: a fixed stratum  $S$  can be perverse or not perverse. The stratum  $S$  is perverse when the action of  $\mathbb{S}^1$  on its link is **not** cohomologically trivial.

In the context of singular actions, the meaning of “Euler class” it is not clear: there are non trivial circle actions having a contractible orbit space  $B$ . This Euler class  $e$  can be recovered by using the de Rham intersection cohomology  $\mathbb{H}^*(-)$ . It has been proved that  $e$  lives in  $\mathbb{H}_{\bar{e}}^2(B)$  where the Euler perversity  $\bar{e}$  takes the following values

$$\bar{e}(S) = \begin{cases} 0 & \text{when } S \text{ mobile stratum,} \\ 1 & \text{when } S \text{ not perverse fixed stratum} \\ 2 & \text{when } S \text{ perverse stratum} \end{cases}$$

(cf. [1, 5.7]). Notice that the Euler class contains the geometrical information about the nature of the strata.

The main result of this work is the following: the orbit space  $B$  of a modelled action and the Euler class  $e \in \mathbb{H}_{\bar{e}}^2(B)$  determine the intersection cohomology of  $X$  (cf. Corollary 3.3), the real homotopy type of  $X$  (cf. Corollary 3.4) and the perverse real homotopy type of  $X$  (cf. Corollary 3.5). The main tool we use is the Gysin sequence constructed for  $\Phi$  in [1].

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# 1 Intersection cohomology of modelled actions

We recall in this Section the main results of [1] we are going to use in this work.

**1.1 Modelled actions.** A reasonable action of the circle on a stratified pseudomanifold must produce a stratified pseudomanifold as orbit space. These are the  $\mathbb{S}^1$ -pseudomanifolds of [2, Section 4]. In this work we shall use a variant of this concept, the modelled action  $\Phi: \mathbb{S}^1 \times X \rightarrow X$  of the circle  $\mathbb{S}^1$  on an unfolded pseudomanifold  $X$ , since the unfolded pseudomanifolds support the (de Rham) intersection cohomology (cf. [1]). We list below the main properties of a modelled action  $\Phi: \mathbb{S}^1 \times X \rightarrow X$  of the circle  $\mathbb{S}^1$  on an unfolded pseudomanifold  $X$ . We denote by  $B = X/\mathbb{S}^1$  the orbit space and by  $\pi: X \rightarrow B$  the canonical projection.

(MA.i) *The isotropy subgroup  $\mathbb{S}_x^1$  is the same for each  $x \in S$ . It will be denoted by  $\mathbb{S}_S^1$ .*

(MA.ii) *For each regular stratum  $R$  we have  $\mathbb{S}_R^1 = \{1\}$ .*

(MA.iii) *For each singular stratum  $S$  with  $\mathbb{S}_S^1 = \mathbb{S}^1$ , the action  $\Phi$  induces a modelled action  $\Phi_{L_S}: \mathbb{S}^1 \times L_S \rightarrow L_S$ , where  $L_S$  is the link of  $S$ .*

(MA.iv) *The orbit space  $B$  is an unfolded pseudomanifold, relatively to the stratification  $\mathcal{S}_B = \{\pi(S) \mid S \in \mathcal{S}_X\}$ , and the projection  $\pi: X \rightarrow B$  is an unfolded morphism.*

(MA.v) *The assignment  $S \mapsto \pi(S)$  induces the bijection  $\pi_{\mathcal{S}}: \mathcal{S}_X \rightarrow \mathcal{S}_B$ .*

The action  $\Phi$  classifies the strata of  $X$  in two types: the stratum  $S$  is *mobile* when  $\mathbb{S}_S^1$  is finite and it is *fixed* when  $\mathbb{S}_S^1 = \mathbb{S}^1$ . In this work, we need another classification for the fixed strata. A fixed stratum  $S$  is *perverse* when  $H^*(L_S \setminus \Sigma_{L_S}) \neq H^*((L_S \setminus \Sigma_{L_S})/\mathbb{S}^1) \otimes H^*(\mathbb{S}^1)$ , where  $\Sigma_{L_S}$  is the singular part of the link  $L_S$  (cf. [1, 5.6 (3)]).

**1.2 Examples.** Consider  $B = c\mathbb{S}^2$ . Essentially, there are three different modelled actions having  $B$  as the orbit space.

$$\Phi_1: \mathbb{S}^1 \times c\mathbb{S}^3 \longrightarrow c\mathbb{S}^3 \quad \text{defined by} \quad \Phi_1(z, [(u, v), t]) = [(z \cdot u, z \cdot v), t],$$

$$\Phi_2: \mathbb{S}^1 \times c(\mathbb{S}^2 \times \mathbb{S}^1) \longrightarrow c(\mathbb{S}^2 \times \mathbb{S}^1) \quad \text{defined by} \quad \Phi_2(z, [(x, w), t]) = [(x, z \cdot w), t], \text{ and}$$

$$\Phi_3: \mathbb{S}^1 \times (c(\mathbb{S}^2) \times \mathbb{S}^1) \longrightarrow (c(\mathbb{S}^2) \times \mathbb{S}^1) \quad \text{defined by} \quad \Phi_3(z, ([x, t], w)) = ([x, t], z \cdot w).$$

The difference between these actions lies on the geometrical nature of the singular stratum  $\{\vartheta\}$  (vertex) of  $B$ . In fact, in the first case the stratum  $\{\vartheta\}$  comes from a perverse stratum, in the second case the stratum  $\{\vartheta\}$  comes from a non-perverse fixed stratum and in the third case the stratum  $\{\vartheta\}$  comes from a mobile stratum.

**1.3 Gysin sequence.** Since the Lie group  $\mathbb{S}^1$  is connected and compact, the subcomplex of the invariant perverse forms computes the intersection cohomology<sup>1</sup> of  $X$ . In fact, for any perversity  $\bar{p}$ , the inclusion  $\left(\Omega_{\bar{p}}^*(X)\right)^{\mathbb{S}^1} \hookrightarrow \Omega_{\bar{p}}^*(X)$  induces an isomorphism in cohomology. This complex can be described in terms of basic data as follows. Consider the graded complex

$$(1) \quad I\Omega_{\bar{p}}^*(X) = \left\{ (\alpha, \beta) \in \Pi^*(B) \oplus \Omega_{\bar{p}-\bar{x}}^{*-1}(B) \mid \begin{array}{l} \|\alpha\|_{\pi(S)} \leq \bar{p}(S) \\ \|d\alpha + (-1)^{|\beta|}\beta \wedge \epsilon\|_{\pi(S)} \leq \bar{p}(S) \end{array} \right\} \text{ if } S \in \mathcal{S}_X^{sing}$$

endowed with the differential  $D(\alpha, \beta) = (d\alpha + (-1)^{|\beta|}\beta \wedge \epsilon, d\beta)$ . Here  $|\cdot|$  stands for the degree of the form,  $\epsilon \in \Pi^2(B)$  is an Euler form and  $\bar{x}$  is the characteristic perversity defined by  $\bar{x}(\pi(S)) =$

<sup>1</sup>For the notions related with the intersection cohomology, we refer the reader to [4, Section 3].

$\begin{cases} 1 & \text{if } S \text{ is a fixed stratum} \\ 0 & \text{if } S \text{ is a mobile stratum} \end{cases}$ . The assignment  $(\alpha, \beta) \mapsto \pi^* \alpha + \pi^* \beta \wedge \chi$  establishes a differential graded isomorphism between  $I\Omega_{\bar{p}}(X)$  and  $(\Omega_{\bar{p}}^*(X))^{\mathbb{S}^1}$ .

From (1) we have the short exact sequence

$$0 \longrightarrow \Omega_{\bar{p}}^*(B) \xrightarrow{\pi_{\bar{p}}} I\Omega_{\bar{p}}^*(X) \xrightarrow{\mathcal{J}_{\bar{p}}} \mathcal{G}_{\bar{p}}^{*-1}(B) \longrightarrow 0,$$

where

- The Gysin term  $\mathcal{G}_{\bar{p}}^{*-1}(B)$  is the differential complex

$$\left\{ \beta \in \Omega_{\bar{p}-\bar{e}}^{*-1}(B) \mid \exists \alpha \in \Pi^*(B) \text{ with } \begin{bmatrix} \|\alpha\|_{\pi(S)} \leq \bar{p}(S) \text{ and} \\ \|d\alpha + (-1)^{|\beta|} \beta \wedge \epsilon\|_{\pi(S)} \leq \bar{p}(S) \end{bmatrix} \text{ if } S \in \mathcal{S}_X^{sing} \right\},$$

- $\mathcal{J}_{\bar{p}}(\alpha, \beta) = \beta$ , and
- $\pi_{\bar{p}}(\omega) = \pi^* \omega$ .

The associated long exact sequence

$$\dots \longrightarrow \mathbb{H}_{\bar{p}}^{i+1}(X) \xrightarrow{\mathcal{J}_{\bar{p}}} H^i(\mathcal{G}_{\bar{p}}^*(B)) \xrightarrow{\mathbf{e}_{\bar{p}}} \mathbb{H}_{\bar{p}}^{i+2}(B) \xrightarrow{\pi_{\bar{p}}} \mathbb{H}_{\bar{p}}^{i+2}(X) \longrightarrow \dots,$$

where  $\mathbf{e}_{\bar{p}}[\beta] = [d\alpha + (-1)^{|\beta|} \beta \wedge \epsilon]$ , is the Gysin sequence.

Recall that the Euler perversity  $\bar{e}$  is defined by  $\bar{e}(S) = \begin{cases} 0 & \text{when } S \text{ mobile stratum,} \\ 1 & \text{when } S \text{ not perverse fixed stratum} \\ 2 & \text{when } S \text{ perverse stratum} \end{cases}$

So, the Euler class  $e = [\epsilon]$  belongs to  $\mathbb{H}_{\bar{e}}^2(B)$ . This class detects the perverse strata: a fixed stratum is perverse iff the Euler class  $e_S \in \mathbb{H}_{\bar{e}}^2(L_S/\mathbb{S}^1)$  of the action  $\Phi_{L_S}: \mathbb{S}^1 \times L_S \rightarrow M_S$ , **does not vanish** (see (MA.iii)). In the next Section, we shall use the following Lemma

**Lemma 1.3.1** *Let  $\bar{p}$  be a perversity with  $\bar{p} \geq \bar{e}$ . If  $X$  is connected and normal, then*

$$(2) \quad H^0(\mathcal{G}_{\bar{p}}^*(B)) \cong \mathbb{R} \quad \text{and} \quad \mathbf{e}_{\bar{p}}(1) = e.$$

*Proof.* Condition  $\bar{p} \geq \bar{e}$  implies  $1 \in \mathcal{G}_{\bar{p}}^*(B)$ . Since  $X$  is connected and normal, then the regular part  $B \setminus \Sigma_B$  is connected. Then  $H^0(\mathcal{G}_{\bar{p}}^*(B)) \cong \mathbb{R}$ . Finally, the definition of  $\mathbf{e}_{\bar{p}}$  gives  $\mathbf{e}_{\bar{p}}(1) = [\epsilon] = e$ .  $\clubsuit$

## 2 Perverse algebras

Although the intersection cohomology  $\mathbb{H}_{\bar{p}}^*(X)$  is not an algebra, we recover this structure by considering all the perversities together. These are the perverse algebras we present in this Section.

**2.1 Perverse algebras.** A *perverse set* is a triple  $(\mathcal{P}, +, \leq)$  where  $(\mathcal{P}, +)$  is an abelian semi-group with an unity element  $\bar{0}$  and  $(\mathcal{P}, \leq)$  is a poset verifying the compatibility condition:

$$\bar{p} \leq \bar{q} \text{ and } \bar{p}' \leq \bar{q}' \implies \bar{p} + \bar{p}' \leq \bar{q}' + \bar{q}', \quad \text{for } \bar{p}, \bar{q}, \bar{p}', \bar{q}' \in \mathcal{P}.$$

In order to simplify the writing, we shall say that  $\mathcal{P}$  is a perverse set.

A *dgc perverse algebra* (or simply a perverse algebra) is a quadruple  $\mathbf{E} = (E, \iota, \wedge, d)$  where

- $E = \bigoplus_{\bar{p} \in \mathcal{P}} E_{\bar{p}}$  where each  $E_{\bar{p}}$  is a graded (over  $\mathbb{Z}$ ) vector space,
- $\iota = \{\iota_{\bar{p}, \bar{q}}: E_{\bar{p}} \rightarrow E_{\bar{q}} \mid \bar{p} \leq \bar{q}\}$  is a family of graded linear morphisms, and
- $(E, d, \wedge)$  is a dgc algebra,

verifying

$$\begin{aligned}
+ \iota_{\bar{p}, \bar{p}} &= \text{Identity} & + \iota_{\bar{q}, \bar{r}} \circ \iota_{\bar{p}, \bar{q}} &= \iota_{\bar{p}, \bar{r}} & + \wedge \left( E_{\bar{p}} \times E_{\bar{p}'} \right) &\subset E_{\bar{p} + \bar{p}'} \\
+ d(E_{\bar{p}}) &\subset E_{\bar{p}} & + \iota_{\bar{p} + \bar{p}', \bar{q} + \bar{q}'}(a \wedge a') &= \iota_{\bar{p}, \bar{q}}(a) \wedge \iota_{\bar{p}', \bar{q}'}(a') & + d \circ \iota_{\bar{p}, \bar{q}} &= \iota_{\bar{p}, \bar{q}} \circ d
\end{aligned}$$

Here,  $\bar{p} \leq \bar{q} \leq \bar{r}$ ,  $\bar{p}' \leq \bar{q}'$ ,  $a \in E_{\bar{p}}$  and  $a' \in E_{\bar{p}'}$ .

Associated to a dgc perverse algebra  $\mathbf{E} = (E, \iota, \wedge, d)$  we have another dgc perverse algebra, namely, its cohomology  $\mathbf{H}(\mathbf{E}) = \left( \bigoplus_{\bar{p} \in \mathcal{P}} H(E_{\bar{p}}, d), \iota, \wedge, 0 \right)$ , where  $\iota$  and  $\wedge$  are induced by the previous  $\iota$  and  $\wedge$ .

A *dgc perverse morphism* (or simply *perverse morphism*)  $\mathbf{f}$  between two perverse algebras  $\mathbf{E} = (E, \iota, \wedge, d)$  and  $\mathbf{E}' = (E', \iota', \wedge', d')$  is given by a family  $\mathbf{f} = \{f_{\bar{p}}: E_{\bar{p}} \rightarrow E_{\bar{p}}'\}$  of differential graded morphisms verifying

$$(3) \quad \iota'_{\bar{p}, \bar{q}} \circ f_{\bar{p}} = f_{\bar{q}} \circ \iota_{\bar{p}, \bar{q}}$$

and

$$(4) \quad f_{\bar{p} + \bar{p}'}(a \wedge b) = f_{\bar{p}}(a) \wedge f_{\bar{p}'}(b).$$

Here,  $\bar{p} \leq \bar{q}$ ,  $a \in E_{\bar{p}}$  and  $b \in E_{\bar{p}'}$ . We shall denote the perverse morphism by  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{E}'$ . It induces the perverse morphism  $\mathbf{f}: \mathbf{H}(\mathbf{E}) \rightarrow \mathbf{H}(\mathbf{E}')$ , defined by  $f_{\bar{p}}[a] = [f_{\bar{p}}(a)]$  for each  $\bar{p}$  and  $[a] \in H(E_{\bar{p}}, d)$ .

When each  $f_{\bar{p}}$  is an isomorphism, we shall say that  $\mathbf{f}$  is a *dgc perverse isomorphism* (or simply *perverse isomorphism*). It induces the perverse isomorphism  $\mathbf{f}: \mathbf{H}(\mathbf{E}) \rightarrow \mathbf{H}(\mathbf{E}')$ .

**2.2 Perverse algebras and modelled actions.** Fix  $\Phi: \mathbb{S}^1 \times X \rightarrow X$  a modelled action. The family of perversities  $\mathcal{P}_X$  of  $X$  has a partial order  $\leq$  and an abelian law  $+$  in such a way that  $\mathcal{P}_X$  is a perverse set. In the same way,  $\mathcal{P}_B$  is a perverse set. Since the two posets  $\mathcal{S}_B^{\text{sing}}$  and  $\mathcal{S}_X^{\text{sing}}$  are isomorphic (cf. (MA.v)), then the perverse sets  $\mathcal{P}_B$  and  $\mathcal{P}_X$  are isomorphic through the map  $\bar{p} \mapsto \bar{p} \circ \pi$  (cf. (MA.iv)). In the sequel, we shall identify these two perverse sets.

Associated to the modelled action  $\Phi$ , we have the following dgc perverse algebras.

$$\begin{aligned}
+ \text{ The } \textit{perverse de Rham algebra}: \mathbf{\Omega}(X) &= \left( \Omega(X) = \bigoplus_{\bar{p} \in \mathcal{P}_X} \Omega_{\bar{p}}(X), \iota, \wedge, d \right). \\
+ \text{ The } \textit{intersection cohomology algebra}: \mathbf{IH}(X) &= \left( \mathcal{IH}(X) = \bigoplus_{\bar{p} \in \mathcal{P}_X} \mathcal{IH}_{\bar{p}}(X), \iota, \wedge, 0 \right).
\end{aligned}$$

Analogously for  $B$ . The quadruple  $\mathbf{I\Omega}(X) = \left( I\Omega(X) = \bigoplus_{\bar{p} \in \mathcal{P}_X} I\Omega_{\bar{p}}(X), \iota, \wedge, D \right)$  is a also perverse algebra.

Here, the wedge product is defined by  $(\alpha, \beta) \wedge (\alpha', \beta') = (\alpha \wedge \alpha', (-1)^{|\alpha'|} \beta \wedge \alpha' + \alpha \wedge \beta')$ . A straightforward calculation shows that the operator

$$(5) \quad \Delta = \{\Delta_{\bar{p}}\}: \mathbf{I\Omega}(X) \rightarrow \mathbf{\Omega}(X),$$

defined by  $\Delta_{\bar{p}}(\alpha, \beta) = \pi^* \alpha + \pi^* \beta \wedge \chi$ , induces a perverse isomorphism in cohomology.

For each perversity  $\bar{p}$  we have the linear morphism  $\rho_{\bar{p}}: \Omega_{\bar{p}}^*(B) \rightarrow I\Omega_{\bar{p}}^*(X)$  defined by  $\rho_{\bar{p}}(\alpha) = (\alpha, 0)$ . The operator  $\rho = \{\rho_{\bar{p}}\}: \Omega(B) \rightarrow I\Omega(X)$  is a perverse morphism. It induces the perverse morphism  $\pi = \Delta \circ \rho: \mathbf{H}(B) \rightarrow \mathbf{H}(X)$ .

### 3 Cohomological classification of modelled actions

We considered in this Section a modelled action  $\Phi: \mathbb{S}^1 \times X \rightarrow X$  whose orbit space is a fixed unfolded pseudomanifold  $B$ . We prove that the intersection cohomology algebra and the (perverse) real homotopy type of  $X$  are determined by the Euler class.

**3.1 Fixing the orbit space.** Consider  $\Phi_1: \mathbb{S}^1 \times X_1 \rightarrow X_1$  and  $\Phi_2: \mathbb{S}^1 \times X_2 \rightarrow X_2$  two modelled actions and write  $B_1$  and  $B_2$  the two orbit spaces. Consider  $f: B_1 \rightarrow B_2$  an unfolded isomorphism. The two posets  $\mathcal{S}_{B_1}^{sing}$  and  $\mathcal{S}_{B_2}^{sing}$  are isomorphic through the map  $\pi_1(S) \mapsto f(\pi_1(S))$ . The perverse sets  $\mathcal{P}_{B_1}$  and  $\mathcal{P}_{B_2}$  are isomorphic through the map  $\bar{p} \mapsto \bar{p} \circ f^{-1}$ . In the sequel, we shall identify this two perverse sets in order to compare the perverse de Rham algebras of  $X_1$  and  $X_2$ .

The induced map  $f^*: \Pi^*(B_2) \rightarrow \Pi^*(B_1)$  is a well defined differential graded isomorphism. It preserves the perverse degree. For each perversity  $\bar{p}$  we write  $f_{\bar{p}}: \Omega_{\bar{p}}^*(B_2) \rightarrow \Omega_{\bar{p}}^*(B_1)$  the differential graded isomorphism defined by  $f_{\bar{p}}(\alpha) = f^* \alpha$ . The operator  $\mathbf{f} = \{f_{\bar{p}}\}: \Omega(B_2) \rightarrow I\Omega(B_1)$ , is a perverse isomorphism. It induces the perverse isomorphism  $\mathbf{f}: \mathbf{H}(B_2) \rightarrow \mathbf{H}(B_1)$ .

The unfolded isomorphism  $f$  is *optimal* when it preserves the nature of the strata, that is, when it sends the fixed (resp. perverse, resp. non-perverse) strata into fixed (resp. perverse, resp. non-perverse) strata. In this case, the two Euler perversities are equal:  $\bar{e}_1(\pi_1(S)) = \bar{e}_2(f(\pi_1(S)))$  for each singular stratum  $S \in \mathcal{S}_{X_1}^{sing}$ . We shall write  $\bar{e}$  for this Euler perversity.

Now we can compare the two Euler classes  $e_1 \in \mathbf{H}_{\bar{e}}^2(B_1)$  and  $e_2 \in \mathbf{H}_{\bar{e}}^2(B_2)$ . We shall say that  $e_1$  and  $e_2$  are *proportional* if there exists a number  $\lambda \in \mathbb{R} \setminus \{0\}$  such that  $f_{\bar{e}}(e_2) = \lambda \cdot e_1$ . As we are going to see, this is the key test for the comparison between the de Rham algebras of  $X_1$  and  $X_2$ .

Finally, we say that the actions  $\Phi_1$  and  $\Phi_2$  *have a common orbit space* if there exists an optimal isomorphism between their orbit spaces.

The three main results of this work come from this Proposition.

**Proposition 3.2** *Let  $X_1, X_2$  be two connected normal unfolded pseudomanifolds. Consider two modelled actions  $\Phi_1: \mathbb{S}^1 \times X_1 \rightarrow X_1$  and  $\Phi_2: \mathbb{S}^1 \times X_2 \rightarrow X_2$ . Let us suppose that there exists an unfolded isomorphism  $f: B_1 \rightarrow B_2$  between the associated orbit spaces. Then, the two following statements are equivalent:*

- (a) *The isomorphism  $f$  is optimal and the Euler classes  $e_1$  and  $e_2$  are proportional.*
- (b) *There exists a perverse isomorphism  $\mathbf{F}: \mathbf{H}(X_2) \rightarrow \mathbf{H}(X_1)$  verifying  $\mathbf{F} \circ \pi_2 = \pi_1 \circ \mathbf{f}$ .*

*Proof.* We proceed in two steps.

(a)  $\Rightarrow$  (b) Since the isomorphism  $f$  is optimal then  $\bar{x}_1 = \bar{x}_2$  and we will denote by  $\bar{x}$  this perversity. Since  $f^* e_2 = f^* [e_2] = \lambda \cdot e_1 = \lambda \cdot [e_1]$ , with  $\lambda \in \mathbb{R} \setminus \{0\}$ , then there exists  $\gamma \in \Omega_{\bar{e}}^1(B_2)$  with  $f^* e_2 = \lambda \cdot e_1 - d(f^* \gamma)$ . For each perversity  $\bar{p}$  we define  $F_{\bar{p}}: I\Omega_{\bar{p}}^*(X_2) \rightarrow I\Omega_{\bar{p}}^*(X_1)$  by

$$F_{\bar{p}}(\alpha, \beta) = (f^*(\alpha - \beta \wedge \gamma), f^*(\lambda \cdot \beta)).$$

The map  $F_{\bar{p}}$  is a well defined differential graded morphism. Let us see that. For each  $(\alpha, \beta) \in I\Omega_{\bar{p}}^*(X_2)$  and for each  $S \in \mathcal{S}_{X_1}^{sing}$  we have

$$- f^*(\alpha - \beta \wedge \gamma) \in \Pi^*(B_1).$$

- $f^*(\lambda \cdot \beta) \in \Omega_{\overline{p}-\overline{x}}^{*-1}(B_1)$ .
- $\|f^*(\alpha - \beta \wedge \gamma)\|_{\pi(S)} = \|\alpha - \beta \wedge \gamma\|_{\pi(f(S))} \leq \max(\|\alpha\|_{\pi(f(S))}, \|\beta\|_{\pi(f(S))} + \|\gamma\|_{\pi(f(S))})$   
 $\leq \max(\overline{p}(S), \overline{p}(S) - \overline{x}(S) + \|\gamma\|_{\pi(f(S))}) \leq \overline{p}(S)$  since  $\|\gamma\|_{\pi(f(S))} \leq \overline{x}(S)$ .
- $\|f^*d(\alpha - \beta \wedge \gamma) + (-1)^{|\beta|} f^*(\lambda \cdot \beta) \wedge \epsilon_1\|_{\pi(S)} = \|f^*d\alpha - f^*(d\beta \wedge \gamma) - (-1)^{|\beta|} f^*(\beta \wedge d\gamma) + (-1)^{|\beta|} f^*(\beta \wedge \epsilon_2) + (-1)^{|\beta|} f^*(\beta \wedge d\gamma)\|_{\pi(S)} = \|f^*(d\alpha + (-1)^{|\beta|} \beta \wedge \epsilon_2) - f^*(d\beta \wedge \gamma)\|_{\pi(S)} \leq \max(\|d\alpha + (-1)^{|\beta|} \beta \wedge \epsilon_2\|_{\pi(f(S))}, \|d\beta \wedge \gamma\|_{\pi(f(S))}) \leq \overline{p}(S)$ .
- $D_1 F_{\overline{p}}(\alpha, \beta) = (f^*d(\alpha - \beta \wedge \gamma) + (-1)^{|\beta|} f^*(\lambda \cdot \beta) \wedge \epsilon_1, f^*(\lambda \cdot d\beta)) = (f^*(d\alpha + (-1)^{|\beta|} \beta \wedge \epsilon_2) - f^*(d\beta \wedge \gamma), f^*(\lambda \cdot d\beta)) = F_{\overline{p}}(d\alpha + (-1)^{|\beta|} \beta \wedge \epsilon_2, d\beta) = F_{\overline{p}} D_2(\alpha, \beta)$ .

The family  $\mathbf{F} = \{F_{\overline{p}}\}: \mathbf{I}\Omega(X_2) \rightarrow \mathbf{I}\Omega(X_1)$  is a perverse morphism since:

(3) A straightforward calculation.

(4) Consider  $(\alpha, \beta) \in I\Omega_{\overline{p}}^*(X_2)$  and  $(\alpha', \beta') \in I\Omega_{\overline{p}'}^*(X_2)$ . Then

$$F_{\overline{p}+\overline{p}'}((\alpha, \beta) \wedge (\alpha', \beta')) = F_{\overline{p}+\overline{p}'}((\alpha \wedge \alpha', (-1)^{|\alpha'|} \beta \wedge \alpha' + \alpha \wedge \beta')) = (f^*(\alpha \wedge \alpha' - (-1)^{|\alpha'|} \beta \wedge \alpha' \wedge \gamma - \alpha \wedge \beta' \wedge \gamma), f^*((-1)^{|\alpha'|} \lambda \cdot \beta \wedge \alpha' + \lambda \cdot \alpha \wedge \beta')) = (f^*(\alpha - \beta \wedge \gamma), f^*(\lambda \cdot \beta)) \wedge (f^*(\alpha' - \beta' \wedge \gamma), f^*(\lambda \beta')) = F_{\overline{p}}(\alpha, \beta) \wedge F_{\overline{p}'}(\alpha', \beta').$$

In fact, the perverse morphism  $\mathbf{F}$  is a perverse isomorphism, the inverse is given by  $\mathbf{F}^{-1} = \{F_{\overline{p}}^{-1}\}$ , where  $F_{\overline{p}}^{-1}(\alpha, \beta) = (f^{-*}\alpha + \lambda^{-1} \cdot f^{-*}\beta \wedge \gamma, \lambda^{-1} \cdot f^{-*}\beta)$ . We conclude that the induced operator  $\mathbf{F}: \mathbf{H}(X_2) \rightarrow \mathbf{H}(X_1)$  is a perverse isomorphism. Finally, the equality  $\mathbf{F} \circ \pi_2 = \pi_1 \circ \mathbf{f}$  comes from

$$F_{\overline{p}}((\pi_2)_{\overline{p}}(\alpha)) = F_{\overline{p}}(\alpha, 0) = (f^*\alpha, 0) = (\pi_1)_{\overline{p}}(f^*\alpha) = (\pi_1)_{\overline{p}}(f_{\overline{p}}(\alpha)),$$

where  $\overline{p}$  is a perversity and  $\alpha \in I\Omega_{\overline{p}}^*(B_2)$ .

(b)  $\Rightarrow$  (a) Write  $\mathbf{f} = \{f_{\overline{p}}: \mathbf{H}_{\overline{p}}^*(B_2) \rightarrow \mathbf{H}_{\overline{p}}^*(B_1)\}$  and  $\mathbf{F} = \{F_{\overline{p}}: \mathbf{H}_{\overline{p}}^*(X_2) \rightarrow \mathbf{H}_{\overline{p}}^*(X_1)\}$ . Consider now the Gysin sequences associated to the action  $\Phi_1$  and  $\Phi_2$ . The two Gysin terms are written  ${}_1\mathcal{G}$  and  ${}_2\mathcal{G}$  respectively. Since  $F_{\overline{e}_2} \circ (\pi_2)_{\overline{e}_2} = (\pi_1)_{\overline{e}_2} \circ f_{\overline{e}_2}$  we can construct a commutative diagram

$$(6) \quad \begin{array}{ccccccc} \mathbf{H}_{\overline{e}_2}^1(B_2) & \xrightarrow{(\pi_2)_{\overline{e}_2}} & \mathbf{H}_{\overline{e}_2}^1(X_2) & \xrightarrow{(f_2)_{\overline{e}_2}} & H^0({}_2\mathcal{G}_{\overline{e}_2}^*(B_2)) & \xrightarrow{(\mathbf{e}_2)_{\overline{e}_2}} & \mathbf{H}_{\overline{e}_2}^2(B_2) \xrightarrow{(\pi_2)_{\overline{e}_2}} \mathbf{H}_{\overline{e}_2}^2(X_2) \\ f_{\overline{e}_2} \downarrow & & F_{\overline{e}_2} \downarrow & & \ell \downarrow & & f_{\overline{e}_2} \downarrow \quad F_{\overline{e}_2} \downarrow \\ \mathbf{H}_{\overline{e}_2}^1(B_1) & \xrightarrow{(\pi_1)_{\overline{e}_2}} & \mathbf{H}_{\overline{e}_2}^1(X_1) & \xrightarrow{(f_1)_{\overline{e}_2}} & H^0({}_1\mathcal{G}_{\overline{e}_2}^*(B_1)) & \xrightarrow{(\mathbf{e}_1)_{\overline{e}_2}} & \mathbf{H}_{\overline{e}_2}^2(B_1) \xrightarrow{(\pi_1)_{\overline{e}_2}} \mathbf{H}_{\overline{e}_2}^2(X_1), \end{array}$$

where  $\ell: H^0({}_2\mathcal{G}_{\overline{e}_2}^*(B_2)) \rightarrow H^0({}_1\mathcal{G}_{\overline{e}_2}^*(B_1))$  is an isomorphism. From (2) we get that  $H^0({}_2\mathcal{G}_{\overline{e}_2}^*(B_2))$  is  $\mathbb{R}$  (the constant functions) and therefore  $\ell$  is the multiplication by a number  $\lambda \in \mathbb{R} \setminus \{0\}$ . We prove (a) in two steps.

1. If the isomorphism  $f$  is optimal then the Euler classes  $e_1$  and  $e_2$  are proportional. We have  $\overline{e}_1 = \overline{e}_2 = \overline{e}$ . The formula (2) and the diagram (6) give

$$\lambda \cdot e_1 = \lambda \cdot (\mathbf{e}_1)_{\overline{e}}(1) = f_{\overline{e}}((\mathbf{e}_2)_{\overline{e}}(1)) = f_{\overline{e}}(e_2).$$

2. The isomorphism  $f$  is optimal. It suffices to prove that  $\overline{e}_1(\pi_1(S)) = \overline{e}_2(f(\pi_1(S)))$  for each  $S \in \mathcal{S}_{X_1}^{sing}$ . Since  $H^0({}_1\mathcal{G}_{\overline{e}_2}^*(B_1)) = \mathbb{R}$  then  $1 \in {}_1\mathcal{G}_{\overline{e}_2}^*(B_1)$  and we get that  $\overline{e}_2 - \overline{x}_1 \geq 0$ . So,  $\overline{e}_1(\pi_1(S)) = 0$  if  $\overline{e}_2(f(\pi_1(S))) = 0$ . By symmetry :  $\overline{e}_1(\pi_1(S)) = 0 \iff \overline{e}_2(f(\pi_1(S))) = 0$ .

The fixed strata are the same for both actions. If the perverse strata are different, then we can find a fixed stratum  $S$  with  $\bar{e}_1(\pi_1(S)) \neq \bar{e}_2(f(\pi_1(S)))$  and  $\bar{e}_1(\pi_1(S')) = \bar{e}_2(f(\pi_1(S')))$  for each singular stratum  $S'$  with  $S \preceq S'$ . In particular, the fixed strata and the perverse strata are the same on  $L_S$ . We have proved that the Euler classes of the actions  $\Phi_{1,L_S}: \mathbb{S}^1 \times L_S \rightarrow L_S$  and  $\Phi_{2,L_S}: \mathbb{S}^1 \times L_S \rightarrow L_S$  are proportional through a non-vanishing factor. So, they vanish or not simultaneously. This would give  $\bar{e}_1(\pi_1(S)) = \bar{e}_2(f(\pi_1(S)))$  (cf. 1.3). Contradiction. ♣

**3.2.1 Remark.** The connectedness and the normality of  $X_1$  and  $X_2$  have only been used in the proof of  $(b) \Rightarrow (a)$ .

The first result of this work shows how the Euler class of the action determines the intersection cohomology algebra of the unfolded pseudomanifold  $X$ .

**Corollary 3.3** *Consider two modelled actions  $\Phi_1: \mathbb{S}^1 \times X_1 \rightarrow X_1$  and  $\Phi_2: \mathbb{S}^1 \times X_2 \rightarrow X_2$  having a common orbit space. If the Euler classes  $e_1$  and  $e_2$  are proportional then intersection cohomology algebra of  $X_1$  and  $X_2$  are isomorphic.*

The second result of this work shows how the Euler class of the action determines the real homotopy type of the stratified unfolded  $X$ .

**Corollary 3.4** *Let  $X_1, X_2$  be two connected normal unfolded pseudomanifolds. Consider two modelled actions  $\Phi_1: \mathbb{S}^1 \times X_1 \rightarrow X_1$  and  $\Phi_2: \mathbb{S}^1 \times X_2 \rightarrow X_2$  having a common orbit space. If the two Euler classes  $e_1$  and  $e_2$  are proportional then the real homotopy type of  $X_1$  and  $X_2$  are the same.*

*Proof.* The real homotopy type of  $X_k$  is determined by the dgca  $\Omega_{\bar{0}}^*(X_k)$  for  $k = 1, 2$  (cf. [3]). The result comes from the following sequence of dgca quasi-isomorphisms:

$$\Omega_{\bar{0}}^*(X_2) \xleftarrow{\Delta_{2,\bar{0}}} I\Omega_{\bar{0}}^*(X_2) \xrightarrow{F_{\bar{0}}} I\Omega_{\bar{0}}^*(X_1) \xrightarrow{\Delta_{1,\bar{0}}} \Omega_{\bar{0}}^*(X_1)$$

(cf. (5), Proposition 3.2). ♣

Inspired by the notion of real homotopy type we can define the perverse real homotopy type of an unfolded pseudomanifold in the following way. Two unfolded pseudomanifolds  $X_1$  and  $X_2$  have the same *perverse real homotopy type* if there exists a finite family of perverse quasi-isomorphisms

$$X_1 \leftarrow \bullet \rightarrow \cdots \leftarrow \bullet \rightarrow X_2.$$

Here, a perverse quasi-isomorphism is a perverse isomorphism inducing an isomorphism in cohomology. Notice that, in the Proposition 3.2, we have proved in fact the following result:

**Corollary 3.5** *Consider two modelled actions  $\Phi_1: \mathbb{S}^1 \times X_1 \rightarrow X_1$  and  $\Phi_2: \mathbb{S}^1 \times X_2 \rightarrow X_2$  having a common orbit space. If the two Euler classes  $e_1$  and  $e_2$  are proportional then the perverse real homotopy type of  $X_1$  and  $X_2$  are the same.*

## References

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